Énoncé : $\int_V (\hat{r}\cdot\vec{r}\,')\vec{J}(\vec{r}\,')\,d\tau' = \frac{1}{2}\left\{\int_V \left[\vec{r}\,'\times\vec{J}(\vec{r}\,')\right]d\tau'\right\}\times\hat{r}$

Preuve en 5 étapes :

<u>1ère étape</u>

$$\left(\vec{r}\,'\times\vec{J}\right)\times\hat{r} = -\frac{1}{r}\left\{\vec{r}\times\left(\vec{r}\,'\times\vec{J}\right)\right\}$$

<u>2ième étape</u>

$$\begin{aligned}\vec{r}\times\left(\vec{r}\,'\times\vec{J}\right) &= \vec{r}\,'\left(\vec{r}\cdot\vec{J}\right) - \vec{J}\left(\vec{r}\cdot\vec{r}\,'\right)\\ &= \sum_i\sum_j\left[x_j J_j x_i'\hat{x}_i - x_j x_j' J_i\hat{x}_i\right]\\ &= \sum_i \hat{x}_i\left[\sum_j x_j\left(x_i' J_j - x_j' J_i\right)\right]\end{aligned}$$

<u>3ième étape</u>

$$\int_V x_i' J_j\,d\tau' = \int_V x_i'\vec{J}\cdot\vec{\nabla}' x_j'\,d\tau' = \int_V x_i'\left[\vec{\nabla}\cdot\left(x_j'\vec{J}\right) - \cancel{x_i' x_j'\vec{\nabla}\cdot\vec{J}}\right]d\tau'$$

$$= \int_V\left[\vec{\nabla}\cdot\left(x_i' x_j'\vec{J}\right) - x_j'\vec{J}\cdot\vec{\nabla}x_i'\right]d\tau' = \cancel{\oint_S x_i' x_j'\vec{J}\cdot d\vec{a}} - \int_V x_j'\vec{J}\cdot\vec{\nabla}x_i'\,d\tau'$$

$$\Rightarrow \int_V x_i' J_j\,d\tau' = -\int_V x_j' J_i\,d\tau'$$

<u>4ième étape</u>

$$\begin{aligned}\sum_i\hat{x}_i\sum_j x_j\int_V\left(x_i' J_j - x_j' J_i\right)d\tau' &= -2\sum_i\hat{x}_i\sum_j x_j\int_V x_j' J_i\,d\tau'\\ &= -2\int_V\left(\vec{r}\cdot\vec{r}\,'\right)\vec{J}(\vec{r}\,')\,d\tau'\end{aligned}$$

<u>5ième étape</u>

$$\begin{aligned}\frac{1}{2}\left\{\int_V\left[\vec{r}\,'\times\vec{J}(\vec{r}\,')\right]d\tau'\right\}\times\hat{r} &= -\frac{1}{2r}\int_V \vec{r}\times\left[\vec{r}\,'\times\vec{J}(\vec{r}\,')\right]d\tau'\\ &= -\frac{1}{2r}\left\{-2\int_V\left(\vec{r}\cdot\vec{r}\,'\right)\vec{J}(\vec{r}\,')\,d\tau'\right\}\\ &= \int_V\left(\vec{r}\cdot\vec{r}\,'\right)\vec{J}(\vec{r}\,')\,d\tau'\end{aligned}$$

CQFD